We are now ready to state the main result of this section.

Theorem 6.5 (Cauchy–Goursat theorem) *Let f be analytic in a simply connected domain D. If C is a simple closed contour that lies in D, then*

$$\int_C f(z)\,dz = 0.$$

We give two proofs. The first, by Augustin Cauchy, is more intuitive but requires the additional hypothesis that f' is continuous.

Proof (Cauchy's proof of Theorem 6.5.) If we suppose that f' is continuous, then with C oriented positively we use Equation (6-16) to write

$$\int_C f(z)\,dz = \int_C u\ dx - v\ dy + i\int_C v\ dx + u\ dy. \tag{6-28}$$

If we use Green's theorem on the real part of the right side of Equation (6-28) (with $P = u$ and $Q = -v$), we obtain

$$\int_C u\ dx - v\ dy = \iint_R (-v_x - u_y)\,dx\ dy, \tag{6-29}$$

where R is the region that is the interior of C. If we use Green's theorem on the imaginary part, we get

$$\int_C v\ dx + u\ dy = \iint_R (u_x - v_y)\,dx\ dy. \tag{6-30}$$

If we use the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ in Equations (6-29) and (6-30), Equation (6-28) becomes

$$\int_C f(z)\,dz = \iint_R 0\ dx\ dy + i\iint_R 0\ dx\ dy = 0,$$

and the proof is complete.

In 1883, Edward Goursat (1858–1936) produced a proof that does not require the continuity of f'.

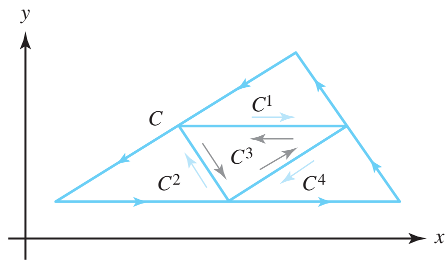

Figure 6.20 The triangular contours C and C^1, C^2, C^3, and C^4.

Proof (Goursat's proof of Theorem 6.5) We first establish the result for a triangular contour C with positive orientation. We then construct four positively oriented contours C^1, C^2, C^3, and C^4 that are the triangles obtained by joining the midpoints of the sides of C, as shown in Figure 6.20.

Each contour is positively oriented, so if we sum the integrals along the four triangular contours, the integrals along the segments interior to C cancel out in pairs, giving

$$\int_C f(z)\,dz = \sum_{k=1}^{4} \int_{C^k} f(z)\,dz. \tag{6-31}$$

Let C_1 be selected from C^1, C^2, C^3, and C^4 so that the following holds:

$$\left|\int_C f(z)\,dz\right| \le \sum_{k=1}^{4} \left|\int_{C^k} f(z)\,dz\right| \le 4\left|\int_{C_1} f(z)\,dz\right|.$$

Proceeding inductively, we carry out a similar subdivision process to obtain a sequence of triangular contours $\{C_n\}$, where the interior of C_{n+1} lies in the interior of C_n and the following inequality holds:

$$\left|\int_{C_n} f(z)\,dz\right| \le 4\left|\int_{C_{n+1}} f(z)\,dz\right|, \qquad \text{for } n = 1, 2, \ldots. \tag{6-32}$$

We let T_n denote the closed region that consists of C_n and its interior. The length of the sides of C_n go to zero as $n \to \infty$, so there exists a unique point z_0 that belongs to all the closed triangular regions $\{T_n\}$. Since D is simply connected, $z_0 \in D$, so f is analytic at the point z_0. Thus, there exists a function $\eta(z)$ such that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0), \tag{6-33}$$

where $\lim_{z \to z_0} \eta(z) = 0$.

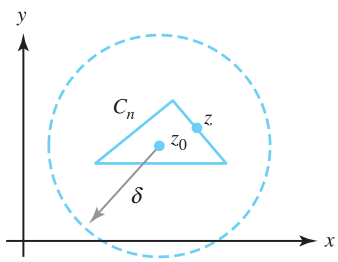

Figure 6.21 The contour C_n that lies in the neighborhood $|z - z_0| < \delta$.

Using Equation (6-33) and integrating f along C_n, we get

$$\begin{aligned}\int_{C_n} f(z)\,dz &= \int_{C_n} f(z_0)\,dz + \int_{C_n} f'(z_0)(z - z_0)\,dz \\ &\quad + \int_{C_n} \eta(z)(z - z_0)\,dz \\ &= [f(z_0) - f'(z_0)\,z_0] \int_{C_n} 1\,dz + f'(z_0) \int_{C_n} z\,dz \\ &\quad + \int_{C_n} \eta(z)(z - z_0)\,dz \\ &= \int_{C_n} \eta(z)(z - z_0)\,dz.\end{aligned}$$

Since $\lim_{z \to z_0} \eta(z) = 0$, we know that given $\varepsilon > 0$, we can find $\delta > 0$ such that

$$|z - z_0| < \delta \quad \text{implies that} \quad |\eta(z)| < \frac{2}{L^2}\varepsilon, \tag{6-34}$$

where L is the length of the original contour C. We can now choose an integer n so that C_n lies in the neighborhood $|z - z_0| < \delta$, as shown in Figure 6.21. Since the distance between any point z on a triangle and a point z_0 interior to the triangle is less than half the perimeter of the triangle, it follows that

$$|z - z_0| < \frac{1}{2} L_n, \qquad \text{for all } z \text{ on } C_n,$$

where L_n is the length of the triangle C_n. From the preceding construction process, it follows that

$$L_n = \left(\frac{1}{2}\right)^n L \quad \text{and} \quad |z - z_0| < \left(\frac{1}{2}\right)^{n+1} L, \qquad \text{for } z \text{ on } C_n. \tag{6-35}$$

We can use Equations (6-32), (6-34), and (6-35) and Theorem 6.3 to conclude

$$\begin{aligned}\left|\int_C f(z)\,dz\right| &\le 4^n \int_{C_n} |\eta(z)(z-z_0)|\,|dz| \\ &\le 4^n \int_{C_n} \frac{2}{L^2}\varepsilon \left(\frac{1}{2}\right)^{n+1} L\,|dz| \\ &= \frac{2^n\varepsilon}{L}\int_{C_n} |dz| \\ &= \frac{2^n\varepsilon}{L}\left(\frac{1}{2}\right)^{n} L = \varepsilon.\end{aligned}$$

Because ε was arbitrary, it follows that our theorem holds for the triangular contour C. If C is a polygonal contour, then we can add interior edges until the interior is subdivided into a finite number of triangles. The integral around each triangle is zero, and the sum of all these integrals equals the integral around the polygonal contour C. Therefore, our theorem also holds for polygonal contours. The proof for an arbitrary simple closed contour is established by approximating the contour "sufficiently close" with a polygonal contour. We omit the details of this last step.